Locality and Bell's inequality

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Abstract

We prove that the locality condition is irrelevant to Bell in equality. We check that the real origin of the Bell's inequality is the assumption of applicability of classical (Kolmogorovian) probability theory to quantum mechanics.

Inequalities among numbers

Lemma (1) For any two numbers $a, c \in [-1, 1]$ the following equivalent inequalities hold:

$$|a \pm c| \leq 1 \pm ac \tag{1}$$

Moreover equality in (1) holds if and only if either $a = \pm 1$ or $c = \pm 1$.

Proof. The equivalence of the two inequalities (1) follows from the fact that one is obtained from the other by changing the sign of c and c is arbitrary in $[-1, 1]$.

Since for any $a, c \in [-1, 1]$, $1 \pm ac \geq 0$, (1) is equivalent to

$$|a \pm c|^2 = a^2 + c^2 \pm 2ac \leq (1 \pm ac)^2 = 1 + a^2c^2 \pm 2ac$$

and this is equivalent to

$$a^2(1 - c^2) + c^2 \leq 1$$

which is identically satisfied because $1 - c^2 \geq 0$ and therefore

$$a^2(1 - c^2) + c^2 \leq 1 - c^2 + c^2 = 1 \tag{2}$$

Notice that in (2) equality holds if and only if $a^2 = 1$ i.e. $a = \pm 1$. Since, exchanging a and c in (1) the inequality remains unchanged, the thesis follows.

Corollary (2) For any three numbers $a, b, c \in [-1, 1]$ the following equivalent inequalities hold:

$$|ab \pm cb| \leq 1 \pm ac \tag{3}$$

and equality holds if and only if $b = \pm 1$ and either $a = \pm 1$ or $c = \pm 1$.

Proof. For $b \in [-1, 1]$,

$$|ab \pm cb| = |b| \cdot |a \pm c| \leq |a \pm c| \tag{4}$$

so the thesis follows from Lemma (1). In (4) the first equality holds if and only if $b = \pm 1$, so also the second statement follows from Lemma (1).

Lemma (3). For any numbers a, a', b, b', $c \in [-1, 1]$, one has

$$|ab - bc| + |ab' + b'c| \leq 2 \tag{5}$$

$$ab + ab' + a'b' - a'b \leq 2 \tag{6}$$

In (5) equality holds if and only if $b, b', a, c = \pm 1$.

Proof. Because of (3)

$$|ab - cb| \leq 1 - ac \tag{7}$$

$$|ab' - cb'| \leq 1 + ac \tag{8}$$

adding (7) and (8) one finds (5). The left hand side of (6) is $\leq$ than

$$|ab - ba'| + |ab' + b'a'| \tag{9}$$

and replacing a' *by* c, (8) becomes the left hand side of (5). If $b, b' = \pm 1$ and $a = \pm 1$ equality holds in (7) and (8) hence in (5). Conversely, suppose that equality holds in (5) and suppose that either $|b| < 1$ or $|b'| < 1$. Then we arrive to the contradiction

$$2 = |b| \cdot |a - a'| + |b'| \cdot |a + a'| < |a - a'| + |a + a'| \leq (1 - aa') + (1 + aa') = 2 \tag{10}$$

So, if equality holds in (5), we must have $|b| = |b'| = 1$. In this case (5) becomes

$$|a - a'| + |a + a'| = 2 \tag{11}$$

and, if either $|a| < 1$ or $|a'| < 1$, then from Lemma (1) we know that

$$|a - a'| + |a + a'| < (1 - aa') + (1 + aa') = 2$$

so we must also have $a, a' = \pm 1$.

Corollary (4). If $a, a', b, b', c \in \{-1, 1\}$, then the inequalities (1) and (5) are equivalent and equality holds in all of them. However the inequality in (6) may be strict.

Proof. From Lemma (1) we know that the inequalities (1) and (2) are equivalent. From Lemma (3) we know that (1) implies (5). Choosing $b' = a$ in (5), since $a = \pm 1$, Lemma (2) implies that (5) becomes

$$|ab - cb| \leq 1 - ac$$

which is equivalent to (1). (6) is equivalent to

$$a(b + b') + a'(b' - b) \leq 2 \tag{12}$$

In our assumptions either $(b + b')$ or $(b' - b)$ is zero, so (12) is either equivalent to

$$a(b + b') \leq 2$$

or to

$$a'(b' - b) \leq 2$$

and in both cases we can choose a, b, b' or a', b, b' so that the product is negative and the inequality is strict.

The Bell inequality

Corollary (1) (Bell inequality) Let A, B, C, D be random variables defined on the same probability space and with values in the interval $[-1, 1]$. Then the following inequalities hold:

$$E(|AB - BC|) \leq 1 - E(AC) \tag{1}$$

$$E(|AB + BC|) \leq 1 + E(AC) \tag{2}$$

$$E(|AB - BC|) + E(|AD + DC|) \leq 2 \tag{3}$$

where E denotes the expectation value in the probability space of the four variables. Moreover (1) is equivalent to (2) and, if either A or C has values ± 1, then the three inequalities are equivalent.

Proof. Lemma (x.1) implies the inequalities

$$|AB - BC| \leq 1 - AC$$

$$|AB + BC| \leq 1 + AC$$

$$|AB - BC| + |AD + DC| \leq 2$$

from which (1), (2), (3) follow by taking expectation and using the fact that $|E(X)| \leq E(|X|)$. The equivalence is established by the same arguments as in Lemma (x.1).

Remark. Bell's original proof, as well as the almost totality of the available proofs of Bell's inequality, deal only with the case of random variables assuming only the values

$+1$ and -1. The present generalization is not without interest because it dispenses from the assumption that the classical random variables, used to describe quantum observables, have the same set of values of the latter ones: a hidden variable theory is required to reproduce the results of quantum theory only when the hidden parameters are averaged over.

Corollary (3) Let $S^1_a, S^1_c, S^2_b, S^2_d$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and with values in the interval $[-1, +1]$. Then the following inequality holds: n

$$\left|E(S^1_a S^2_b) - E(S^1_c S^2_b)\right| + \left|E(S^1_a S^2_d) + E(S^1_c S^2_d)\right| \leq 2$$

Proof. This is a rephrasing of Corollary (2).

Implications of the Bell's inequalities for the singlet correlations

To apply Bell's inequalities to the singlet correlations, considered in the EPR paradox, it is enough to observe that they imply the following

Lemma (1) In the ordinary three-dimensional euclidean space there exist sets of three, unit length, vectors a, b, c, such that it is not possible to find a probability space $(\Omega, \mathcal{F}, P)$ and six random variables S^j_x $(x = a, b, c,\ j = 1, 2)$ defined on $(\Omega, \mathcal{F}, P)$ and with values in the interval $[-1, +1]$, whose correlations are given by:

$$E(S^1_x \cdot S^2_y) = -x \cdot y \qquad ; \qquad x, y = a, b, c \tag{1}$$

where, if $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ are two three-dimensional vectors, $x \cdot y$ denotes their euclidean scalar product, i.e. the sum $x_1y_1 + x_2y_2 + x_3y_3$.

Remark. The random variables S^j_a, S^j_b, S^j_c should represent the spin of the two particles of a singlet pair along three directions in space. The expression in the right-hand side of (1) is the singlet correlation of two spin or polarization observables, theoretically predicted by quantum theory and experimentally confirmed by the Aspect-type experiments.

Proof. Suppose that, for any choice of the unit vectors $x = a, b, c$ there exist random variables S^j_x as in the statement of the Lemma. Then, using Bell's inequality in the form (x.2) with $A = S^1_a$, $B = S^2_b$, $C = S^1_c$), we obtain

$$\left|E(S^1_a S^2_b) + E(S^2_b S^1_c)\right| \leq 1 + E(S^1_a S^1_c) \tag{2}$$

Now notice that, if $x = y$ is chosen in (1), we obtain

$$E(S^1_x \cdot S^2_x) = -x \cdot x = -\left\|x\right\|^2 = -1 \quad ; \qquad x = a, b, c$$

and, since $|S_x^1 S_x^2| = 1$ this is possible if and only if $S_x^1 = -S_x^2$ $(x = a, b, c)$ P–almost everywhere. Using this (2) becomes equivalent to:

$$\left|E(S_a^1 S_b^2) + E(S_b^2 S_c^1)\right| \leq 1 - E(S_a^1 S_c^2)$$

or, again using (1), to:

$$|a \cdot b + b \cdot c| \leq 1 + a \cdot c \tag{3}$$

If the three vectors a, b, c are chosen to be in the same plane and such that a is perpendicular to c and b lies between a and b, forming an angle θ with a, then the inequality (3) becomes:

$$cos\theta + \sin\theta \leq 1 \qquad ; \quad 0 < \theta < \pi/2 \tag{4}$$

But the maximum of the function of $\theta \longmapsto \sin\theta + \cos\theta$ in the interval $[0, \pi/2]$ is $\sqrt{2}$ (obtained for $\theta = \pi/4$). Therefore, for θ close to $\pi/4$, the left-hand side of (4) will be close to $\sqrt{2}$ which is more that 1. In conclusion, for such a choice of the unit vectors a, b, c, random variables $S_a^1, S_b^2, S_c^1, S_c^2$ as in the statement of the Lemma cannot exist.

Definition (2) A local realistic model for the EPR (singlet) correlations is defined by:

(1) a probability space $(\Omega, \mathcal{F}, P)$
(2) for every unit vector x, in the three-dimensional euclidean space, two random variables S_x^1, S_x^2 defined on Ω and with values in the interval $[-1, +1]$ whose correlations, for any x, y, are given by equation (1).

Corollary (3) If a, b, c are chosen so to violate (4) then a local realistic model for the EPR correlations, in the sense of Definition (2), does not exist.

Proof. Its existence would contradict Lemma (1).

Remark. In the literature one usually distinguishes two types of local realistic models – deterministic and stochastic ones. Both are included in Definition (2): the deterministic models are defined by random variables S_x^j with values in the set$\{-1, +1\}$; while, in the stochastic models, the random variables take values in the interval $[-1, +1]$. The original paper [Be64] was devoted to the deterministic case. Starting from [Be71] several papers have been introduced to justify the stochastic models. We prefer to distinguish the definition of the models from their justification.

In the last section of [Be66] (submitted before [Be64], but published after) Bell briefly describes Bohm hidden variable interpretation of quantum theory underlining its non local character. He then raises the question: *... that there is no proof that any hidden variable account of quantum mechanics must have this extraordinary character ...* and, in a footnote added during the proof corrections, he claims that: *... Since the completion of this paper such a proof has been found [Be64].* In the short Introduction to [Be64], Bell reaffirms the same ideas, namely that the result proven by him in this paper shows that: *... any such [hidden variable] theory which reproduces exactly the quantum mechanical predictions* must have *... a grossly nonlocal structure.*

The proof goes along the following scheme: Bell proves an inequality in which, according to what he says (cf. statement after formula (1) in [Be64]):

... The vital assumption [2] is that the result B for particle 2 *does not depend on the setting a, of the magnet for particle* 1*, nor A on b.*

The paper [2], mentioned in the above statement, is nothing but the Einstein, Podolsky, Rosen paper [EPR35] and the locality issue is further emphasized by the fact that he reports the famous Einstein's statement [Ein49]: *... But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former.*

Stated otherwise: according to Bell, Bell's inequality is a consequence of the locality assumption.

It follows that a theory which violates the above mentioned inequality also violates *... the vital assumption* needed, according to Bell, for its deduction, i.e. locality.

Since the experiments prove the violation of this inequality, Bell concludes that quantum theory does not admit a local completion; in particular quantum mechanics is a nonlocal theory. To use again Bell's words:

the statistical predictions of quantum mechanics are incompatible with separable predetermination ([Be64], p.199). Moreover this incompatibility has to be understood in the sense that: *IN a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, howevere remote. Moreover, the signal involved must propagate instantaneously, ...*

An assumption should be considered "vital" for a theorem if, without it, the theorem cannot be proved.

To favor Bell, let us require much less. Namely let us agree to consider his assumption *vital* if the theorem cannot be proved by taking as its hypothesis the negation of this assumption.

If even this minimal requirement is not satisfied, then we must conclude that the given assumption has nothing to do with the theorem.

THEOREM (1) Denote $\mathcal{M}_1$ ($\mathcal{M}_2$)the space of all possible measurements done on system 1 (2). For each unit vector x in the three dimensional euclidean space ($x \in R^3$, $\mid x \mid= 1$) let be given two random variables S^1_x, S^2_x (spin of particle 1 (2) in direction x), defined on a space Ω with a probability P and with values in the 2–point set $\{+1, -1\}$. Fix 3 of these unit vectors a, b, c and suppose that the corresponding random variables satisfy the following non locality condition [VIOLATING BELL'S *vital assumption*]: suppose that for some function $\overline{S}^1_a, \overline{S}^1_a : \Omega \times \mathcal{M}$,

$$S^1_a = \overline{S}^1_a(\cdot, m_1) \quad (1 \textit{ depends on } 2) \tag{2}$$

$$S^2_a = S^2_a(\cdot, m_2) \quad (2 \textit{ depends on } 1) \tag{3}$$

with $m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2$ and similarly for b and c. [nothing changes in the proof if we add further dependences].

Then the random variables S^1_a, S^2_b, S^1_c satisfy the inequality

$$\mid \langle S^1_a S^2_b\rangle - \langle S^2_b S^1_c\rangle \mid\leq 1 - \langle S^1_a S^1_c\rangle \tag{4}$$

If moreover the anticorrelation condition (assumption R)

$$\langle S^1_x \cdot S^2_x\rangle = -1 \qquad ; \qquad x = a, b, c \tag{5}$$

is also satisfied, then Bell's inequality holds in the form

$$\mid \langle S^1_a S^2_b\rangle - \langle S^2_b S^1_c\rangle \mid\leq 1 + \langle S^1_a S^2_c\rangle \tag{6}$$

PROOF. The random variables $A = S^1_a, B = S^2_b, C = S^1_c$ satisfy the assumptions of Theorem (x.) therefore (1), which in our notations becomes (4), holds. If also condition (5) is satisfied then, since the variables take values in the set $\{-1, +1\}$, with probability 1 one must have $S^1_x = -S^2_x$ ($x = a, b, c$) and therefore $\langle S^1_a S^1_c\rangle = -\langle S^1_a S^2_c\rangle$. Using this identity, (4) becomes (6).

Summing up: Theorem (1) proves that Bell's inequality is satisfied if one takes as hypothesis the negation of his "vital assumption". From this we conclude that Bell's "vital assumption" not only is not "vital" but in fact has nothing to do with Bell's inequality.

The above discussion is not a refutation of the Bell inequality: it is a refutation of Bell's claim that his formulation of locality is an essential assumption for its validity: since the locality assumption is irrelevant for the proof of Bell's inequality it follows that this inequality cannot discriminate between local and non local hidden variable theories, as claimed both in the introduction and the conclusions of Bell's paper.

In particular: Theorem (1) gives an example of situations in which:

(i) Bell's locality condition is violated while his inequality is satisfied.

In a recent experiment with M. Regoli we have produced examples of situations in which:

(ii) Bell's locality condition is satisfied while his inequality is violated.

It is then legitimate to ask:

if Bell's vital assumption is irrelevant for the deduction of Bell's inequality, which is the really vital assumption which guarantees the validity of this inequality?

This natural question was first answered in [Ac81].

The role of the counterfactual argument in Bell's proof

Bell uses the counterfactual argument in an essential way in his proof because it is easy to check that formula (13) in Bell's famous '64 paper is the one which allows him to reduce, in the proof of his inequality, all consideration to the A–variables (forgetting what, in his notations, are the B–variables). The pairs of chameleons (as well as our experiment) provide a counterexample precisely to this fomula.

Clauser, Horne, Shimony, Holt [ClHo69] gave a proof of a variant of the Bell inequality for quadruples $(a,b)(a,b')(a',b)(a',b')$. They start from the following inequality among numbers

$$\mid ab + ab' + a'b - a'b' \mid \leq 2 \tag{1}$$

which, for $a, b, b', a \in [-1,1]$, follows from

$$\mid b + b' \mid + \mid b - b' \mid \leq 2 \tag{2}$$

because

$$\mid ab{+}ab'{+}a'b{-}a'b' \mid = \mid a(b{+}b'){+}a'(b{-}b') \mid \leq \mid a \mid \cdot \mid b{+}b' \mid + \mid a' \mid \cdot \mid b{-}b' \mid \leq \mid b{+}b' \mid + \mid b{-}b' \mid \leq 2$$

The proof of (2) is obvious because it is equivalent to

$$\mid b + b' \mid^2 + \mid b - b' \mid^2 = b^2 + b'^2 + 2bb' + b^2 + b'^2 - 2bb' = 2b^2 + 2b'^2 \leq 4$$

which is identically satisfied.

REMARK (1) Notice that an inequality of the form

$$\mid a_1b_1 + a_2b_2' + a_3'b_3 - a_4'b_4' \mid \leq 2 \tag{1a}$$

would be obviously false. In fact, for example the choice

$$a_1 = b_1 = a_2 = b_2' = a_3' = b_3 = b_4' = 1 \qquad ; \qquad a_4' = -1$$

would give

$$\mid a_1b_1 + a_2b_2' + a_3'b_3 - a_4'b_4' \mid = 4$$

That is: for the validity of (1) it is absolutely essential that the number a is the same in the first and the second term and similarly for a' in the 3–d and the 4–th, b' in the 2–d and the 4–th, b in the first and the 3–d.

This inequality among numbers can be extended to pairs of random variables as follows.

(P1) Instead of four numbers $a, b, b', a \in [-1,1]$, one considers four functions

$$S_a^{(1)}, S_b^{(2)}, S_{a'}^{(1)}, S_{b'}^{(2)}$$

on a space Λ (whose points are called *hidden parameters*) and with values in $[-1,1]$.

(P2) One postulates that there exists a probability measure P on Λ which defines the joint distribution of each of the following four pairs of functions

$$(S_a^{(1)}, S_b^{(2)})(S_a^{(1)}, S_{b'}^{(2)})(S_{a'}^{(1)}, S_b^{(2)})(S_{a'}^{(1)}, S_{b'}^{(2)}) \tag{3}$$

Remark (2) Notice that ($P2$) automatically implies that the joint distributions of the four pairs of functions can be deduced from a joint distribution of the whole quadruple, i.e. the existence of a single Kolmogorov model for these four pairs.

With these premises, for each $\lambda \in \Lambda$ one can apply the inequality (1) to the four numbers

$$S_a^{(1)}(\lambda), S_b^{(2)}(\lambda), S_{a'}^{(1)}(\lambda), S_{b'}^{(2)}(\lambda)$$

and deduce that

$$\mid S_a^{(1)}(\lambda)S_b^{(2)}(\lambda) + S_a^{(1)}(\lambda)S_{b'}^{(2)}(\lambda) + S_{a'}^{(1)}(\lambda)S_b^{(2)}(\lambda) - S_{a'}^{(1)}(\lambda)S_{b'}^{(2)}(\lambda) \mid \leq 2 \tag{4}$$

From this, taking P–averages, one obtains

$$\mid \langle S_a^{(1)}S_b^{(2)}\rangle + \langle S_a^{(1)}S_{b'}^{(2)}\rangle + \langle S_{a'}^{(1)}S_b^{(2)}\rangle - \langle S_{a'}^{(1)}S_{b'}^{(2)}\rangle \mid = \tag{5a}$$

$$\mid \int \Big(S_a^{(1)}(\lambda)S_b^{(2)}(\lambda) + S_a^{(1)}(\lambda)S_{b'}^{(2)}(\lambda) + S_{a'}^{(1)}(\lambda)S_b^{(2)}(\lambda) - S_{a'}^{(1)}(\lambda)S_{b'}^{(2)}(\lambda)\Big) dP(\lambda) \mid \leq \tag{5b}$$

$$\leq \int \mid S_a^{(1)}(\lambda)S_b^{(2)}(\lambda) + S_a^{(1)}(\lambda)S_{b'}^{(2)}(\lambda) + S_{a'}^{(1)}(\lambda)S_b^{(2)}(\lambda) - S_{a'}^{(1)}(\lambda)S_{b'}^{(2)}(\lambda) \mid dP(\lambda) \leq 2 \tag{5c}$$

Remark (3) Notice that in the step from (5a) to (5b) we have used in an essential way the existence of a joint distribution for the whole quadruple, i.e. the fact that all these random variales can be realized in the same probability space.

In EPR type experiments we are interested in the case in which the four pairs $(a,b)(a,b')(a',b)(a',b')$ come from four mutually incompatible experiments. Let us assume that there is a hidden parameter, determining the result of each of these experiments. This means that we interpret the number $S_a^{(1)}(\lambda)$ as the value of the spin of particle 1 in direction a, determined by the hidden parameter λ.

There is obviously no reason to postulate that the hidden parameter, determining the result of the first experiment is exactly the same one which determines the result of the second experiment. However, when CHSH consider the quantity (4), they are implicitly doing the much stronger assumption that the same hidden parameter λ determines the results of all the four experiments. This assumption is quite unreasonable from the physical point of view and in any case it is a much stronger assumption than simply postulating the existence of hidden parameters. The latter assumption would allow CHSH only to consider the expression

$$\mid S_a^{(1)}(\lambda)S_b^{(2)}(\lambda_1) + S_a^{(1)}(\lambda)S_{b'}^{(2)}(\lambda_2) + S_{a'}^{(1)}(\lambda)S_b^{(2)}(\lambda_3) - S_{a'}^{(1)}(\lambda)S_{b'}^{(2)}(\lambda_4) \tag{4}$$

and, as shown in Remark (x.) the maximum of this expression is not 2 but 4 and this does not allow to deduce the Bell inequality.

Contrarily to the original Bell's argument, the CHSH proof of the Bell inequality does not use explicitly the counterfactual argument. Since one can perform experiments also on quadruples, rather than on triples, as originally proposed by Bell, has led some authors to claim that the counterfactual argument is not essential in the deduction of the Bell inequality. However we have just seen in section (x.) that the hidden assumption as in Bell's proof, i.e. the realizability of all the random variales involved in the same probability space, is also present in the CHSH argument. The following lemma shows that, under the singlet assumption, the conclusion of the counterfactual argument follows from the hidden assumption of Bell and of CHSH.

Lemma (4) Suppose that all the random variales in (3) are realized in the same probability space. Then, if the singlet condition:

$$\langle S_x^{(1)} S_x^{(2)} \rangle = -1$$

is satisfied, then the condition

$$S_x^{(1)} = -S_x^{(2)} \tag{5}$$

(i.e. formula (13) in Bell's '64 paper) is true almost everywhere. Proof. We have to prove that, if f and g are random variables defined on a probability space (Λ, P) and with values in $[-1, 1]$, then

$$\langle fg \rangle := \int_\Lambda fg dP = -1$$

if and only if $P(fg = -1) = 1$. If $P(fg > -1) > 0$, then

$$\int_\Lambda fg dP = -P(fg = -1) - \int_{fg > -1} |fg| dP > -P(fg = -1) - P(fg > -1) > -1$$

against our assumption.

Summing up: if you want to compare the predictions of a hidden variable theory with quantum theory in the EPR experiment (so that at least we admit the validity of the singlet law) then the hidden assumption, of realizability of all the random variales in (3) in the same probability space, (without which Bell's inequality cannot be proved) implies the same conclusion of the counterfactual argument. Stated otherwise: the counterfactual argument is implicit when you postulate the singlet condition and the realizability on a single probability space. It does not matter if you use triples or quadruples.

In the CHSH scheme:

$$(a,b)\ ,\quad (a',b')\ ,\quad (a,b')\ ,\quad (a',b')$$

the agreement required by the experimenters is the following:

– 1 will measures the same observable in experiments I and III, and the same observable in experiments II and IV;

– 2 will measure the same observable in experiments I and II, and the same observable in experiments III and IV.

Here there is no restriction a priori on the choice of the observables to be measured.

In the Bell scheme the experimentalists agree that:

– 1 measures the same observable in experiments I and III,

– 2 measures the same observable in experiments I and II

– 1 and 2 choose a priori, i.e. before the experiment begins, a direction c and agree that 1 will measure spin in direction c in experiment II and 2 will measure spin in direction c in experiment III (strong agreement)

The strong agreement can be replaced by the following (weak agreement):

– 1 and 2 choose a priori, i.e. before the experiment begins, a finite set of directions $c_1, \ldots, c_K$ and agree that 1 will measure spin in a direction choosen randomly among the directions $c_1, \ldots, c_K$ in experiment II and 2 will do the same in experiment III

In this scheme there is an a priori restriction on the choice of some of the observables to be measured.

If the directions, fixed a priori in the plane, are K, then the probability of a coincidence, corresponding to a totally random (equiprobable) choice, is

$$P(x_{II}^{(1)} = x_{III}^{(2)}) = \sum_{\alpha=1}^{K} (x_{II}^{(1)} = \alpha ; x_{III}^{(2)} = \alpha) = \sum_{\alpha=1}^{K} \frac{1}{K^2} = \frac{1}{K}$$

This shows that, contrarily than in the CHSH scheme, the choice has to be restricted to a finite number of possibilities oterwise the probability of coincidence will be zero.

Many physicists illude themselves to exorcize the necessity to use the random variables ξ_i's by restricting themselves to results of measurements. This is an illusion because it simply amounts to replace the function $a(\xi_1)$ by the sequence $a_n(\xi_1)$ where the index n labels the result in the given sample and ξ, as before, is an index labeling the experiment (and similarly for the other ones). By a simple change of notations my discussion covers both cases.

So:

if you want to keep the inequality (1) you MUST realize all the random variables in (3) in the same probability space!

ii) the physicists never use statements such as "realizability on a single probability space" but they hide this mathematical assumption in their notations, i.e. they use the same symbol to denote the results of different mutually incompatible experiments (cf. comments before formula (4)).

iii) without the assumption: "realizability on a single probability space" (or one of the equivalent hidden formulations introduced in the physical literature), one cannot prove Bell's inequality.

[ADDITIONAL REMARK: if you remember, in order to be chivalric and to help the opponents of our thesis, we have introduced the operation of "cleaning", which in the case of the inequality (1) amounts to eliminate all the results of the experiments except those for which

$$a(\xi_1) = a(\xi_2)$$

$$a'(\xi_3) = a'(\xi_4)$$

It is easy to prove (this is an exercise in classical probability) that even after the "cleaning" the maximum of expression (4) is 4 and not 2.]

Theorem (x.1) has only two assumptions:

(i) that the random variables take values in the interval $[-1,+1]$

(ii) that the random variables are defined on the same probability space

Since we are dealing with spin variables, assumption (i) is reasonable.

Let us consider assumption (ii). This is equivalent to the claim that the three probability measures P_{ab}, P_{ac}, P_{cb}, representing the distributions of the pairs (S^1_a, S^2_b), (S^1_c, S^2_b), (S^1_a, S^2_c) respectively, can be obtained by restriction from a single probability measure P, representing the distribution of the quadruple $S^1_a, S^1_c, S^2_b, S^2_c$.

This is indeed a strong assumption because, due to the incompatibility of the spin variables along non parallel directions, the three correlations

$$\langle S^1_a S^2_b\rangle \quad , \quad \langle S^1_c S^2_b\rangle \quad , \quad \langle S^1_a S^2_c\rangle \tag{6}$$

can only be estimated in different, in fact mutually incompatible, series of experiments. If we label each series of experiments by the corresponding pair (i.e. $(a,b),(b,c),(c,a)$), then we cannot exclude the possibility that also the probability measure in each series of experiments will depend on the corresponding pair. In other words, each of the measures $P_{a,b}, P_{b,c}, P_{c,a}$ describes the joint statistics of a pair of commuting observables (S^1_a, S^2_b), (S^1_c, S^2_b), (S^1_a, S^2_c) and there is no a priori reason to postulate that all these joint distributions for pairs can be deduced from a single distribution for the quadruple $\{S^1_a, S^1_c, S^2_b, S^2_c\}$.

We have already proved in Theorem (x.1) that this strong assumption implies the validity of the Bell inequality. Now let us prove that it is the truly *vital* assumption for the validity of this inequality, i.e. that, if this assumption is dropped, i.e. if no single distribution for quadruples exist, then it is an easy exercise to construct counterexamples violating Bell's inequality. To this goal one can use the following lemma:

Lemma (1) . Let be given three probability measures P_{ab}, P_{ac}, P_{cb} on a given (measurable) space $(\Omega, \mathcal{F})$ and let $S^1_a, S^1_c, S^2_b, S^2_d$ be functions, defined on $(\Omega, \mathcal{F})$ with values in the interval $[-1,+1]$, and such that the probability measure P_{ab} (resp. P_{cb}, P_{ac}) is the distribution of the pair (S^1_a, S^2_b) (resp. (S^1_c, S^2_b), (S^1_a, S^2_c)) For each pair define the corresponding correlation

$$\kappa_{ab} := \langle S^1_a, S^2_b\rangle := \int S^1_a S^2_b dP_{ab}$$

and suppose that, for $\varepsilon, \varepsilon' = \pm$, the joint probabilities for pairs

$$P^{\varepsilon\varepsilon'}_{x,y} := P(S^1_x = \varepsilon 1 \varepsilon' \ ; \ S^2_y = \varepsilon' 1)$$

satisfy:

$$P^{++}_{xy} = P^{--}_{xy} \qquad ; \qquad P^{+-}_{xy} = P^{-+}_{xy} \tag{1}$$

$$P^{+}_{x} = P^{-}_{x} = 1/2 \tag{2}$$

then the Bell inequality

$$|\kappa_{ab} - \kappa_{bc}| \leq 1 - \kappa_{ac} \tag{3}$$

is equivalent to

$$|P_{ab}^{++} - P_{bc}^{++}| + P_{ac}^{++} \leq \frac{1}{2} \tag{3a}$$

Proof. The inequality (3) is equivalent to

$$|2P_{ab}^{++} - 2P_{ab}^{+-} - 2P_{bc}^{++} + 2P_{bc}^{+-}| \leq 1 - 2P_{ac}^{++} + 2P_{ac}^{+-} \tag{4}$$

Using the identity (equivalent to (2))

$$P_{xy}^{+-} = \frac{1}{2} - P_{xy}^{++} \tag{5}$$

the left hand side of (4) becomes the modulus of

$$2(P_{ab}^{++} - P_{ab}^{+-}) - 2(P_{bc}^{++} - P_{bc}^{+-}) = 2\left(P_{ab}^{++} - \frac{1}{2} + P_{ab}^{++}\right) - 2\left(P_{bc}^{++} - \frac{1}{2} + P_{bc}^{++}\right)$$

$$= 4(P_{ab}^{++} - P_{bc}^{++}) \tag{6}$$

and, again using (5), the right hand side of (4) is equal to

$$1 - 2\left(P_{ac}^{++} - \frac{1}{2} + P_{ac}^{++}\right) = 2 - 4P_{ac}^{++} \tag{7}$$

Summing up, (3) is equivalent to

$$|P_{ab}^{++} - P_{bc}^{++}| \leq \frac{1}{2} - P_{ac}^{++} \tag{8}$$

Corollary (2) . There exist triples of P_{ab}, P_{ac}, P_{cb} on the 4–point space $+1, -1 \times +1, -1$ which satisfy conditions (1), (2) of Lemma (1) and are not compatible with any probability measure P on the 6–point space $+1, -1 \times +1, -1 \times +1, -1$. Proof. Because of conditions (1), (2) the probability measures P_{ab}, P_{ac}, P_{cb} are uniquely determined by the three numbers

$$P_{ab}^{++}, P_{ac}^{++}, P_{cb}^{++} \in [0, 1] \tag{9}$$

Thus, if we choose these three numbers so that the inequality (3a) is not satisfied, the Bellinequality (3) cannot be satisfied because of Lemma (1).

The fact that there is no a priori reason to postulate that the joint distributions of the pairs (S_a^1, S_b^2), (S_c^1, S_b^2), (S_a^1, S_c^2) can be deduced from a single distribution for the quadruple $S_a^1, S_c^1, S_b^2, S_c^2$, does not necessarily mean that such a common joint distribution does not exist.

On the contrary, in several physically meaningful situations, we have good reasons to expect that such a joint distribution should exist even if it might not be accessible to direct experimental verification.

This is a simple consequence of the so–called *hypothesis of realism* which is justified whenever we are entitled to believe that the results of our measurements are *pre–determined*. In the words of Bell: *Since we can predict in advance the result of measuring any chosen component of* σ_2*, by previously measuring the same component of* σ_1*, it follows that the result of any such measurement must actually be predetermined.*

Consider for example a box containing pairs of balls. Suppose that the experiments allow to measure either the color or the weight or the material of which each ball is made of, but the rules of the game are that on each ball only one measurement at a time can be performed. Suppose moreover that the experiments show that, for each property, only two values are realized and that, whenever a simultaneous measurement of the same property on the two elements of a pair is performed, the resulting answers are always discordant. Up to a change of convenction and in appropriate units, we can always suppose that these two values are ± 1 and we shall do so in the following.

Then the joint distributions of pairs (of properties relative to different balls) are accessible to experiment, but those of triples, or quadruples, are not.

Nevertheless, it is reasonable to postulate that, in the box, there is a well defined (although *purely Platonic*, in the sense of not being accessible to experiment) number of balls with each given color, weight and material. These numbers give the relative frequencies of triples of properties for each element of the pair hence, using the perfect anticorrelation, a family of joint probabilities for all the possible sextuples. More precisely, due to the perfect anticorrelation, the relative frequency of the triples of properties

$$[S_a^1 = a_1] \ , \ [S_b^1 = b_1] \ , \ [S_c^1 = c_1]$$

where $a_1, b_1, c_1 = \pm 1$ are equal to the relative frequency of the sextuples of properties

$$[S_a^1 = a_1] \ , \ [S_b^1 = b_1] \ , \ [S_c^1 = c_1] \ , \ [S_a^2 = -a_1] \ , \ [S_b^2 = -b_1] \ , \ [S_c^2 = -c_1]$$

and, since we are confining ourselves to the case of 3 properties and 2 particles, the above ones, when a_1, b_1, c_1 vary in all possible ways in the set $\{\pm 1\}$, are all the possible configurations.

According to the quantum probabilistic interpretation, what Einstein, Podolsky, Rosen, Bell and several other who have discussed this topic, call *the hypothesis of realism* should be called in a more precise way the hypothesis of the *ballot box realism* as opposed to hypothesis of the *chameleon realism*.

The point is that, according to the quantum probabilistic interpretation, the term *predetermined* should not be confused with the term *realized a priori*, which has been discussed in section (x.). The chameleon provides a simple example of this distinction: a chameleon becomes *deterministically* green on a leaf. In this sense we can surely claim that its color on a leaf is *predetermined*. However this does not mean that the chameleon was green also before jumping on the leaf.

The chameleon metaphora describes a mechanism which is *perfectly local*, even *deterministic* and surely *classical and macroscopic*; moreover there are no doubts that the situation it describes is *absolutely realistic*. Yet this realism, being different from the *ballot box realism*, allows to render free from metaphysics statements of the orthodox interpretation such as: *the act of measurement creates the value of the measured observable*.

Finally, and most important for its implications relatively to the EPR argument, the chameleon realism provides a simple and natural counterexample of a situation in which the results are *predetermined* however THE COUNTERFACTUAL ARGUMENT IS NOT APPLICABLE.

Imagine in fact a box in which there are many pairs of chameleons. In each pair there is exactly an healthy one, which becomes green on a leaf and brown on a log, and a mutant one, which becomes brown on a leaf and green on a log; moreover exactly one of the chameleons in each pair weights 100 grams and exactly one 200 grams. A measurement consists in separating the members of each pair, each one in a smaller box, and in performing one and only one measurement on each member of each pair.

Thus, with respect to the observables: *color and weight* the pairs of chameleons behave exactly as EPR pairs: whenever the same observable is measured on both elements of a pair, the results are opposite. However, suppose I measure the color on the leaf, of one element of a pair and the weight of the other one and suppose the answers I find are: *green* and 100 *grams*. Can I conclude that the second element of the pair is *brown and weights* 100 *grams*? Clearly not because there is no reason to believe that the second member of the pair, of which the weight was measured while in a box, was also on a leaf.

One might argue that the correct observables to be considered are not: *color and weight*, but: *color on the leaf, color on the log, and weight*. These are 2–valued observables (because we do not know a priori if we are measuring the healthy or the mutant chameleon). From this point of view the measurement interaction enters the very definition of an observable. However also in this interpretation, which is more similar to the quantum mechanical situation, the counterfactual argument cannot be applied because it amounts to answer *"brown"* to the question: *which is the color on the leaf, if I have measured the weight and if I know that the chameleon is the mutant one?* (this because the measurement of the other one gave green on the leaf). But this answer is not correct, because it could well be that inside the box there is a leaf and the chameleon is interacting with it while I am measuring its weight, but it could also be that it is interacting with a log, also contained inside the box in which case, being a mutant, it would be green.

If this is not the case, then it is trivial to construct counterexamples violating Bell's inequality.

Therefore if we can produce an example of a 2-particle system in which the Heisenberg evolution of each particle's observable satisfies Bell's locality condition, but the Schroedinger evolution of the state , i.e. the expectation value $\rangle \cdot \langle$, depends on the pair (a,b) of measured observables, we can claim that this counterexample abides with the same definition of locality as Bell's theorem.

Bell's inequalities and the chamaleon effect

Definition (1) Let S be a physical system and $\mathcal{O}$ a family of observable quantities relative to this system. We say that the it chamaleon effect is reaallized on S if, for any measurement M of an observable $A \in \mathcal{O}$, the dynamical evolution of S depends on the observable A. If D denotes the state space of S, this means that the change of state from the beginning to the end of the experiment is described by a map

$$T_A \ : \ D \to D$$

Remark. The explicit form of the dependence of T_A on A depends on both the system and the measurement and many concrete examples can be constructed. An example in the quantum domain is discussed in [Ac98] and the experiment of [AcRe99] realizes an example in the classical domain.

Remark If the system S is composed of two sub–systems S_1 and S_2, we can also consider the case in which the evolutions of the two subsystems are different in the sense that, for system 1, we have one form of functional dependence, T^1_A, of the evolution on the observable A and, for system 2, we have another form of functional dependence, T^2_A. In the experiment of [AcRe99], the state space is the unit disk D in the plane, the observables are parametrized by angles in $[0, 2\pi)$ (or equivalently by unit vectors in the unit circle) and, for each observable $S^{(1)}_\alpha$ of system 1

$$T^1_\alpha := R_\alpha$$

and, for each observable $S^{(2)}_\alpha$ of system 2

$$T^2_\alpha := R_{\alpha+\pi}$$

where R_α denotes (counterclockwise) rotation of an angle α.

Let us consider Bell's inequalities by assuming that a chamaleon effect

$$(S^1_a, S^2_b) \mapsto (S^1_a \circ T^1_a, S^2_b \circ T^2_b)$$

is present. Denoting E the common initial state of the composite system $(1,2)$, (e.g. singlet state), the state at the end of the measurement will be

$$E \circ (S_a^1 \circ T_a^1, S_b^2 \circ T_b^2)$$

Now replace $S_x^{(j)}$ by:

$$\tilde{S}_x^{(j)} := S_x^{(j)} \circ T_x^{(j)}$$

Since the $\tilde{S}_x^{(j)}$ take values ± 1, we know from Theorem (x.) that, if we postulate the existence of joint probabilities for the triple $\tilde{S}_a^1, \tilde{S}_b^2, \tilde{S}_c^1$, compatible with the tw correlations $E(\tilde{S}_a^1 \tilde{S}_b^2), E(\tilde{S}_c^1 \tilde{S}_b^2)$, then the inequality

$$|E(\tilde{S}_a^1 \tilde{S}_b^2) - E(\tilde{S}_c^1 \tilde{S}_b^2)| \leq 1 - E(\tilde{S}_a^1 \tilde{S}_c^1)$$

holds and, if we also have the singlet condition

$$E(S_c^1(T_c^1 p) S_c^2(T_c^2 p)) = -1 \tag{1}$$

then a.e.

$$\tilde{S}_c^1 = -\tilde{S}_c^2$$

and we have the Bell's inequality. Thus, if we postulate the same probability space, even the chamaleon effect alone is not sufficient to guarantee violation of the Bell's inequality.

Therefore the fact that the three experiments are done on different and incompatible samples must play a crucial role.

As far as the chameleon effect is concerned, let us notice that, in the above statement of the problem the fact that we use a single initial probability measure E is equivalent to postulate that, at time $t = 0$ the three pairs of observables

$$(S_a^1, S_b^2) \quad , \quad (S_c^1, S_b^2) \quad , \quad (S_a^1, S_c^1)$$

admit a common joint distribution, in fact E.

In this section we show that, combining the chameleon effect with the fact that the three experiments refer to different samples, then even in very simple situations, no cleaning conditions can lead to a proof of the Bell's inequality.

If we try to apply Bell's reasoning to the empirical data, we have to start from the expression

$$\left| \frac{1}{N} \sum_j S_a^1(T_a^1 p_j^I) S_b^2(T_b^2 p_j^I) - \frac{1}{N} \sum_j S_c^1(T_c^1 p_j^{II}) S_b^2(T_b^2 p_j^{II}) \right| \tag{1}$$

which we majorize by

$$\frac{1}{N} \sum_j \left| S_a^1(T_a^1 p_j^I) S_b^2(T_b^2 p_j^I) - S_c^1(T_c^1 p_j^{II}) S_b^2(T_b^2 p_j^{II}) \right| \tag{2}$$

But, if we try to apply the inequality among numbers to the expression

$$\left| S_a^1(T_a^1 p_j^I) S_b^2(T_b^2 p_j^I) - S_c^1(T_c^1 p_j^{II}) S_b^2(T_b^2 p_j^{II}) \right| \tag{3}$$

we see that we are not dealing with the situation covered by Lemma (x.1), i.e.

$$|ab - cb| \leq 1 - ac \tag{4}$$

because, since

$$S_b^2(T_b^2 p_j^I) \neq S_b^2(T_b^2 p_j^{II}) \tag{5}$$

the left hand side of (4) must be replaced by

$$|ab - cb'| \tag{6}$$

whose maximum, for $a, b, c, b' \in [-1, +1]$ is 2 and not $1 - ac$.

Bell's implicit assumption of the single probability space is equivalent to the postulate that, for each $j = 1, \ldots, N$

$$p_j^I = p_j^{II} \tag{3}$$

Physically this means that:

the hidden parameter in the first experiment is the same as the hidden parameter in the second experiment

This is surely a very implausible assumption.

Notice however that, without this assumption, Bell's argument cannot be carried over and we cannot deduce the inequality because we must stop at equation (3).

[Be64] Bell J.S: On the Einstein Podolsky Rosen Paradox Physics 1 no.3. 195-200 1964.

[Be66] Bell J.S.: On the Problem of Hidden Variables in Quantum Mechanics. Rev. Mod. Phys. 38 (1966) 447-452

[ClHo69] J.F. Clauser , M.A. Horne, A. Shimony, R. A. Holt, Phys. Rev. Letters, 49, 1804-1806 (1969); J. S. Bell, *Speakable and unspeakable in quantum mechanics.* (Cambridge Univ. Press, 1987).

[ClHo74] Clauser J.F., Horne M.A.: Experimental Consequences of Objective Local Theories. Physical Review D, vol. 10, no. 2 (1974)

[EPR35] Einstein A., Podolsky B., Rosen N. Can quantum mechanical description of reality be considered complete ? Phys. Rev. 47 (1935) 777-780

[Ein49] A. Einstein in: Albert Einstein: Philosopher Scientist. Edited by P.A. Schilpp, Library of Living Philosophers, Evanston, Illinois, p.85 (1949)